18.702 Comments on Problem Set 1

1. Let $G = D_n$ be the dihedral group of symmetries of an n-gon, with the usual generators and relations: $x^n = 1$, $y^2 = 1$, $yx = x^{-1}y$.

(a) *Let $\rho : G \to GL(V)$ be a representation of G, let v be an eigenvector for ρ_x, and let $w = yv$. Show that the vectors v, w span an invariant subspace of V. Therefore, if ρ is irreducible, its dimension is at most 2.*

(b) *Determine the one-dimensional representations of G.*

Let ρ be a representation of G. Then because $yx = x^{-1}y$, $\rho_y\rho_x = \rho_x^{-1}\rho_y$. If ρ is one-dimensional, then because $GL_1 = \mathbb{C}^*$ is abelian, $\rho_x\rho_y = \rho_y\rho_x = \rho_x^{-1}\rho_y$. Therefore $\rho_x^2 = 1$. We also have $x^n = 1$, so $\rho_x^n = 1$. If n is odd, $\rho_x = 1$, and if n is even, $\rho_x = \pm 1$. And, $\rho_y = \pm 1$ because $y^2 = 1$. There are two one-dimensional representations when n is odd, and four when n is even.

(c) *Show that if ρ is an irreducible representation of dimension 2, the eigenvectors of ρ_x and of ρ_y are distinct.*

If v is an eigenvector of ρ_x and an eigenvector of ρ_y, then v spans an invariant subspace, and then ρ isn't irreducible.

(d) *Determine the isomorphism classes of irreducible representations of dimension 2.*

We start with a basis of eigenvectors for ρ_x. The matrix $R_x$of ρ_x will be diagonal: $R_x = \begin{pmatrix} \lambda & \\ & \lambda' \end{pmatrix}$, where λ, λ' are nth roots of unity. Say that $R_y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $y^2 = 1$, $R_y^2 = I$, but because ρ is irreducible, $R_y \neq I$. Therefore $\operatorname{trace} R_y = a + d = 0$ and $\det R_Y = ad - bc = -1$. The relation $yx = x^{-1}y$ reads $\begin{pmatrix} a\lambda & b\lambda' \\ c\lambda & d\lambda' \end{pmatrix} = \begin{pmatrix} \lambda^{-1}a & \lambda^{-1}b \\ \lambda'^{-1}c & \lambda'^{-1}d \end{pmatrix}$. If $b = c = 0$ both R_x and R_y are diagonal and ρ isn't irreducible. Therefore $\lambda' = \lambda^{-1}$. Then since $\lambda \neq \lambda'$, $a = d = 0$, and $R_y = \begin{pmatrix} & b \\ c & \end{pmatrix}$. Changing our basis changes R_x, R_y by conjugation. Conjugating by $\begin{pmatrix} 1 & \\ & c^{-1} \end{pmatrix}$ leaves R_x unchanged and changes R_y to $R_y' = \begin{pmatrix} & bc \\ 1 & \end{pmatrix}$. Since $\det R_y' = \det R_y = -1$, $bc = -1$. The representation depends only on the choice of the nth root of unity λ.

2. Chapter 10, Exercise 4.2 (a group of order 55)

The Class Equation is $55 = 1 + 5 + 5 + 11 + 11 + 11 + 11$. There are seven irreducible characters, and their dimensions are $1, 1, 1, 1, 1, 5, 5$.

3'. Chapter 10, Exercise 4.10 (completing a character table)

(a) There is one missing character, of dimension 4:

$$\chi_5 \mid 4 \quad -1 \quad 0 \quad 0 \quad 0$$

It is determined by orthogonality. An easy way to determine this character is to use the fact (which we haven't proved) that the columns of the table are orthogonal.

(b) The order of the conjugacy class of an element g determines the order of the centralizer $Z(g)$ by the counting formula (see page 196 of text). Since g centralizes g, the order of g divides $|Z(g)|$ and $|Z(g)|$ is a proper divisor of the order 55 of the group. Thus the order of a is 5, and the orders of b, c, d are either 2 or 4. Since $\chi_4(b) = i$, the order of b is 4. Similarly, the order of c is 4. Then b^2 has order 2. It must be in the remaining class. The order of d is 2.

(c,d,e) The kernel of the representation ρ_2 is a normal subgroup H of order 10. Since C_{10} has only two elements of order 2 while the kernel has 5 such elements, $H = D_5$. If N is any proper normal subgroup, we can take a nontrivial representation of the quotient group: $G/N \to GL(V)$. The composition of maps $G \to G/N \to GL(V)$ will be a representation of G, and N will be in its kernel. So N must be contained in the kernel of one of the representations of G. The only other normal subgroup is the kernel of ρ_3, which is a cyclic group of order 5.

4. Chapter 10, Exercise M.1 (vibrations of a molecule)

Let

$$R_x = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \quad \text{and} \quad R_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the matrices that represent the standard respresentation of the dihedral group G. Then in block form with 2×2 blocks,

$$S_x = \begin{pmatrix} 0 & 0 & R_x \\ R_x & 0 & 0 \\ 0 & R_x & 0 \end{pmatrix} \quad S_y = \begin{pmatrix} R_y & 0 & 0 \\ 0 & 0 & R_y \\ 0 & R_y & 0 \end{pmatrix}$$

The space V decomposes into irreducible representations as follows:

$W_1 : (v_1, v_2, v_3) = (v, v, v)$, v is arbitrary.

This space represents parallel translations of the molecule.

$W_2 : (v_1, v_2, v_3) = c(e_2, R_x e_2, R_x^2 e_2)$,

This space represents rotations.

$W_3 : (v_1, v_2, v_3) = c(e_1, R_x e_1, R_x^2 e_1)$

This space represents vibrations in which the molecule remains equilateral, but stretches and/or shrinks.

W_4: The orthogonal space to $W_1 \oplus W_2 \oplus W_3$.

This space contains the vector $w = c(2, 0, -1, -\sqrt{3}, -1, \sqrt{3})$. It is the span of w and $S_x w$.

Chemists would call these four spaces *modes* of vibration, though they might eliminate the first two, as not being true vibrations.

The fourth mode is confusing. Taking $c > 0$, the vector w represents a vibration in which the atom a_1 moves horizontally to the right, and a_2, a_3 compensate by moving in to the left while coming together. At time $t_0 + \Delta t$, the molecule becomes isoceles.